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Phase of Ising spins on modular networks analogous to social polarization

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Coordination processes in complex systems can be related to the problem of collective ordering in networks, many of which have modular organization. Investigating the order-disorder transition for Ising spins on modular random networks, corresponding to consensus formation in society, we observe two distinct phases: (i) ordering within each module at a critical temperature followed by (ii) global ordering at a lower temperature. This indicates polarization of society into groups having contrary opinions can persist indefinitely even when mutual interactions between agents favor consensus.

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Critical phenomena associated with order-disorder transitions is of central importance in statistical physics [1]. These results also have significant implications for understanding social phenomena where coordination dynamics is observed, e.g., in consensus formation [2,3] and adoption of innovations [4]. Such processes can be analyzed using the spin models of statistical physics which are generic systems for studying cooperative phenomena. The spin orientations correspond to several equivalent but mutually exclusive choices made by an agent on the basis of information about the choice of the majority in its local neighborhood. The simplest case is when an agent decides between two competing choices where the dynamics can be modeled by an Ising system defined on a network reflecting the social contact pattern.

One of the most remarkable features of complex networks seen in many different contexts is their modular organization [5]. Modular networks consist of subnetworks whose nodes have a significantly higher connection density compared to the overall density of the network [6–8]. A recent analysis of a network of mobile phone users, reconstructed according to their call frequency and duration, have shown the existence of such modular structure in the organization of social contacts [9]. This has also been observed in other social networks, e.g., of scientific collaborators [10], electronic mail communication [11,12], the PRETTY-GOOD-PRIVACY (PGP) encryption "web of trust" [13] and even that of nonhuman animals [14]. Study of dynamical processes on such networks [15] can help in understanding how individual behavior at the microscopic level relates to social phenomena at the macroscopic level.

In this Rapid Communication, we investigate a modular network of Ising spins with ferromagnetic interactions and report the existence of a phase with ordering among spins in each module in the absence of global ordering. This modular ordered phase [Fig. 1(a)] is reached from the disordered state of the system through a continuous transition by lowering the temperature to the critical value T_c^m. Further reducing the temperature to T_c^g results in another continuous transition to a state where all the modules are aligned in parallel, i.e., the globally ordered phase [Fig. 1(b)]. Our results help to understand how contrary opinions can coexist in society even when mutual interactions favor consensus. In the social context, the concept of physical temperature corresponds to a measure of noise, e.g., due to imperfect information or uncertainty on the part of the agent, or the existence of idiosyncratic beliefs. Below the critical temperature T_c^g, the relaxation time to global order diverges as the modularity increases. Thus, even when global order is achievable (i.e., $T<T_c^g$), the time required to achieve consensus increases rapidly as the social organization becomes more modular. However, this time can be altered significantly by considering processes that are observed in society. We implement this by (i) introducing an external field modeling positive feedback effects which reinforce the choice adopted by the majority [16] or (ii) varying the strength of couplings between communities relative to those of intramodular links, which is consistent with the well-known "weak-tie" hypothesis for social networks [17].

We consider a modular random network consisting of N nodes arranged into n_m modules, each having $n(=N/n_m)$ nodes [8]. The connection probability between a pair of nodes belonging to the same module is ρ_i, while that for nodes belonging to different modules is ρ_o. The modularity of the network can be changed continuously by varying the ratio of intermodular to intramodular connectivity, $r=\frac{\rho_o}{\rho_i} \in [0,1]$, keeping the average degree $\langle k \rangle$ constant. For $r=0$, the network gets fragmented into a set of isolated clusters, while at $r=1$, it becomes a homogeneous or Erdos-Renyi random network. On placing Ising spins on the nodes of the network, the Hamiltonian for the system is given by

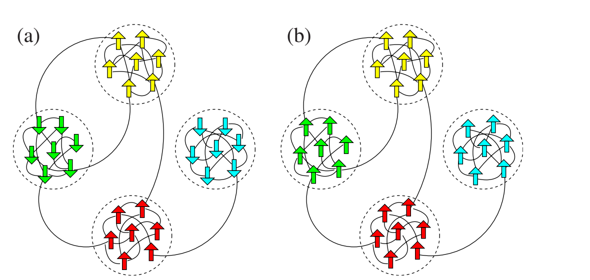

FIG. 1. (Color online) Ordering in modular networks: (a) local ordering within modules without global order and (b) the globally ordered state.

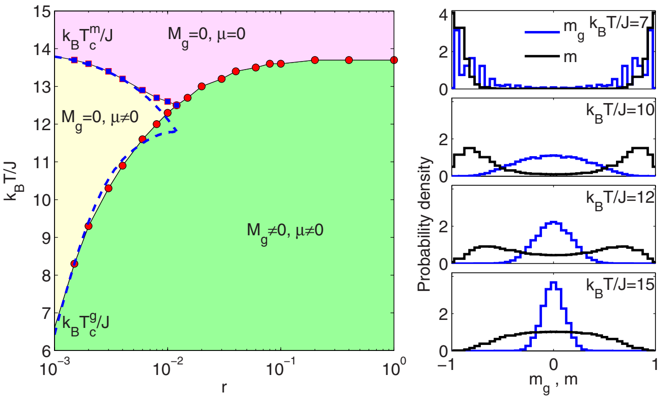

FIG. 2. (Color online) Critical behavior of Ising model on modular network. (a) The $r-k_BT/J$ phase diagram indicating the existence of three distinct phases: disordered ($M_g=\mu=0$), modular order ($M_g=0,\ \mu\neq0$), and global order ($M_g\neq0,\ \mu\neq0$). The solid lines are phase boundaries obtained from Monte Carlo simulation, while the broken lines indicate the critical temperatures T_c^m and T_c^g obtained from analytical expressions given in the text. (b) The distributions for global ($m_g=M_g/N$) and modular ($m=\mu/n$) magnetizations per site at different temperatures showing the two phase transitions for $r=0.002$. At low temperature ($T=7J/k_B$) both curves have bimodal nature, indicating global order. As T increases to $10J/k_B$ ($>T_c^g$), the global order disappears resulting in unimodal distribution of m_g. For temperatures higher than T_c^m ($T=15J/k_B$), both curves are unimodal as the system is disordered. All results shown are for $N=512$, $n_m=16$, and $\langle k\rangle=14$.

$$H=-\sum_{i,j}J_{ij}\sigma_i\sigma_j, \tag{1}$$

where $\sigma_i=\pm1$ is the Ising spin on the ith node and the coupling strength J_{ij} is $J(>0)$ if i,j are connected and zero otherwise. The positive value of J (ferromagnetic coupling) implies that each link tries to align the two spins connected by it.

Starting from a disordered state, we have performed Monte Carlo simulation for updating the spin states using the Wolff algorithm [18]. For any configuration, we can measure the average magnetic moment per module, $\mu=\langle|\Sigma_{i\in s}\sigma_i|\rangle$, the averaging being over all modules $s=1,\ldots,n_m$, and the total (or global) magnetization of the network, $M_g=\Sigma_i\sigma_i$. At equilibrium, the resulting phase diagram (Fig. 2) clearly shows the existence of three phases, corresponding to (a) the disordered state ($\mu=0,\ M_g=0$), (b) a globally ordered state ($\mu\neq0,\ M_g\neq0$), and (c) a state with only modular order ($\mu\neq0,\ M_g=0$). At low r, as the temperature is decreased the system undergoes two transitions: first, at $T=T_c^m$, from the disordered state to the state with modular order, and next, at $T=T_c^g$, to the globally ordered state, where all spins in the network are aligned in parallel. As the modularity of the network decreases with increasing r, the two critical temperatures T_c^m and T_c^g approach each other and finally converge.

The phase diagram can be reproduced analytically by considering the free energy for the system. The magnetic moment of a single module is $\mu=n(2f_+-1)$, where f_+ is the fraction of "up" ($\sigma=+1$) spins in that module. We assume that, at equilibrium, the magnetic moment of each of the n_m modules has the same magnitude, with a fraction f_+^m being positive ($+\mu$) and the remainder being negative ($-\mu$). This is valid in the regime of *strong modularity*, i.e., $r\ll1$. Thus, the total magnetic moment for the system is $M_g=n_m\mu(2f_+^m-1)$. The contribution to the internal energy of the system from each module is $U_i=-JL_i(2f_+-1)^2$, where $L_i=\frac{n(n-1)}{2}\rho_i$ is the number of links within a module. This is based on the mean-field assumption that the neighborhood of all spins are identical. To obtain the internal energy contribution for interactions between modules, U_o, we note that each of the modules can be treated as "spins" of moment μ with $L_o=\frac{n_m(n_m-1)}{2}\rho_o$ links between them. Analogous to the preceding expression for U_i, we get $U_o=-JL_o\mu^2(2f_+^m-1)^2$. Then, the free energy for the system is

$$F(f_+,f_+^m)=n_m(U_i-TS_i)+U_o-TS_o, \tag{2}$$

where the entropy terms $S_i=-nk_B[f_+\log(f_+)+(1-f_+)\log(1-f_+)]$ and $S_o=-n_mk_B[f_+^m\log(f_+^m)+(1-f_+^m)\log(1-f_+^m)]$ correspond, respectively, to the different ways in which nf_+ up spins can be distributed among n spins within each module, and $n_mf_+^m$ modules with moment $+\mu$ can be distributed among n_m modules.

To analyze the critical behavior of the system we minimize the free energy F [Eq. (2)] with respect to f_+ giving

$$\frac{1}{4f_+-2}\log\frac{f_+}{1-f_+}=\frac{T_c^m}{T}+\frac{J\rho_on(n_m-1)(2f_+^m-1)^2}{k_B}, \tag{3}$$

where $T_c^m=2JL_i/(nk_B)$. This indicates a continuous phase transition at the modular critical temperature T_c^m, below which spins within a module are ordered but $f_+^m=1/2$. This state corresponds to the phase for which $\mu\neq0,\ M_g=0$. As the temperature decreases below T_c^m, the different modules get aligned with each other at a temperature T_c^g. This is obtained by minimizing F with respect to f_+^m:

$$\frac{1}{2(2f_+^m-1)}\log\frac{f_+^m}{1-f_+^m}=\frac{T_c^g}{T}, \tag{4}$$

which shows a continuous phase transition at the global critical temperature $T_c^g=J\rho_o(n_m-1)n^2(2f_+-1)^2/k_B$. The expression for T_c^g does not have a closed analytic form and it is obtained by numerical minimization. As $r\to1$, the network loses its modular structure and becomes a homogeneous random network so that Eqs. (3) and (4) give the critical temperature $T_c^m=T_c^g=J\langle k\rangle/k_B$.

The above analysis for finite-size networks are valid even in the *thermodynamic limit* when it is approached by increasing the number of modules, n_m. Note that, if we instead increased the number of nodes in a module, n (keeping n_m fixed), the modularity of the network is lost and hence there is only a single continuous order-disorder transition at $T_c=J\langle k\rangle/k_B$. The two types of ordering seen for the Ising model on a modular network imply that not only can consensus formation in a society be an extremely slow process [19], but under certain conditions it may never be achieved and communities with contrary opinions can persist indefinitely.

We now look at how the time required to reach the equilibrium state corresponding to global order varies with the

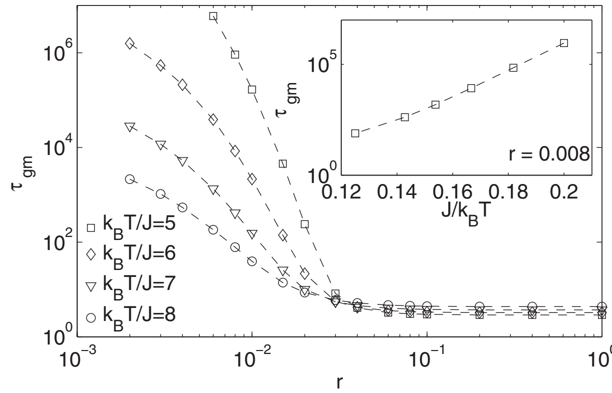

FIG. 3. Relaxation time to the globally ordered state, τ_{gm}, as a function of r at different temperatures. With increasing modularity, τ_{gm} diverges. (Inset) The variation in τ_{gm} with $1/T$ at $r=0.008$. For all cases, $N=512$, $n_m=16$, and $\langle k\rangle=14$.

modularity of the network. Starting from an initially disordered state ($M_g=0,\ \ \mu=0$), spins are updated using Glauber dynamics [20]. Figure 3 shows that the relaxation to global order takes an extremely long time as the system become more modular. To understand this, we first consider a single *isolated* module of n nodes with L_i intramodular links. Under the mean-field approximation, the time evolution of the magnetization per site $m=\mu/n$ is described by

$$\frac{dm(t)}{dt}+m(t)=\tanh\left[\frac{T_c}{T}m(t)\right], \tag{5}$$

where $T_c=2JL_i/(nk_B)$. At equilibrium, this reduces to the usual mean-field equation, $m=\tanh(mT_c/T)$, indicating that T_c is the critical temperature. Defining *relaxation time* τ to be the time in which m becomes 0.5, we obtain from Eq. (5)

$$\tau=\int_{0^+}^{0.5}\frac{dx}{\tanh(xT_c/T)-x}. \tag{6}$$

Although it is small at low temperatures, τ diverges as $T\rightarrow T_c$ due to critical slowing down.

In a network consisting of several modules, the relaxation time to modular order, τ_{mm}, is identical to τ derived above with $T_c=T_c^m$. To obtain the relaxation time to global order, we first note that the free energy of a single module $F_m(f_+)=U_i-TS_i$ has two minima at f_+^0 and $1-f_+^0$ (say). These correspond to the magnetic moments $\pm\mu$, which are separated by a free-energy maximum at $f_+=1/2$ corresponding to $\mu=0$. To switch from $+\mu$ to $-\mu$ (or vice versa), a module has to overcome an energy barrier $\Delta=F_m(1/2)-F_m(f_+^0)$. Thus, the time required to attain a global magnetization M_g is slowed down by the factor $\exp[\Delta/(k_BT)]$. Defining the global relaxation time, τ_{gm}, to be the time in which the global magnetization per site $M_g/(n_m\mu)$ becomes 0.5, we obtain from Eq. (6)

$$\tau_{gm}=\exp\left(\frac{\Delta}{k_BT}\right)\int_{0^+}^{0.5/|2f_+^0-1|}\frac{dx}{\tanh(xT_c^g/T)-x}. \tag{7}$$

Here we have assumed that $\tau_{gm}\gg\tau_{mm}$, which is valid for $T\ll T_c^g$ and low r. In this region, τ_{gm} diverges with increasing modularity of the network, the trend agreeing with the numerical results of Fig. 3. Note that, at low temperatures ($T\ll T_c^g$), the integral in Eq. (7) can be neglected and $\tau_{gm}\approx\exp(\Delta/k_BT)\propto\exp(J\rho_i/k_BT)$, assuming $f_+^0\approx 1$. Thus, the relaxation to global order becomes exponentially slower as the temperature decreases (Fig. 3, inset). The above results indicate that even when $T<T_c^g$, a strongly modular network may require an extremely long time to reach the globally ordered state, which explains why previous studies may have erroneously observed a separate phase without global order in this region [21].

It may appear from the preceding analysis that achieving global consensus is extremely difficult in a real social network having modular organization. However, we now discuss possible mechanisms by which the time to attain global order can be changed significantly. First, we look at the role of an external magnetic field which is proportional to the global magnetization M_g. This corresponds to positive feedback effects in social systems, where, although two competing alternatives are initially equivalent, as more and more agents switch to one particular option, it becomes the preferred choice [16]. Introducing such a field, the Hamiltonian for the system becomes $H=\Sigma_{i,j}J_{ij}\sigma_i\sigma_j-h(\Sigma_i\sigma_i)^2$. The external field h has no effect in the absence of global order, but when M_g is nonzero, the field drives the system to the equilibrium state corresponding to global order much faster as seen in Fig. 4(a). The free energy in this case can be obtained by replacing JL_m with $JL_m+hn_m^2$ in Eq. (2). Thus, the field effectively increases the intermodular interactions (making the network less modular) which drives the system away

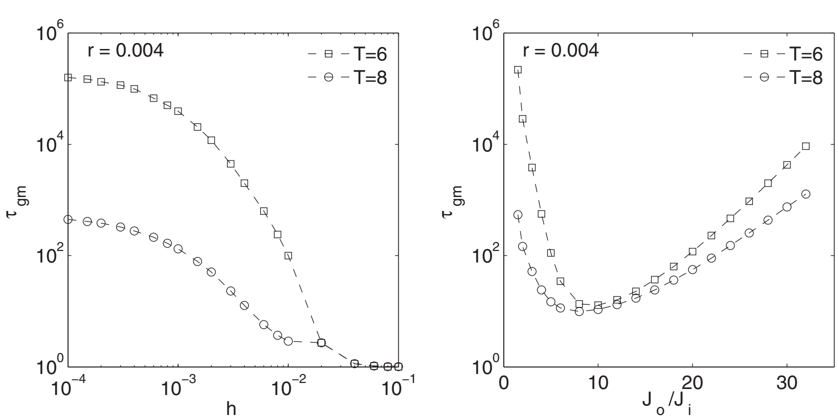

FIG. 4. The relaxation time to the globally ordered state, τ_{gm}, as a function of (a) the external field h and (b) the ratio of inter to intramodular coupling strength J_o/J_i at different temperatures. The minima of τ_{gm} occur when $J_o/J_i\simeq\langle k_{in}\rangle$, the average number of links a node has with other nodes in its module. For all cases, $N=512$, $n_m=16$, $\langle k\rangle=14$, and $r=0.004$.

from the critical point by increasing T_c^g, thereby reducing τ_{gm}.

We next consider the possibility that the interaction strengths for intermodular connections (J_o) may be different from those of intramodular links (J_i). From Eq. (2), it is clear that increasing the ratio J_o/J_i is equivalent to increasing r. Therefore, as the intramodular links increase in strength relative to the intermodular links (making the network even more modular), the time to achieve global order (τ_{gm}) increases. On the other hand, when the intermodular links are stronger, τ_{gm} decreases up to a point and then, with increasing J_o, starts increasing as the nodes in different communities get strongly coupled thereby destroying the identity of individual modules [Fig. 4(b)]. This *nonmonotonic* behavior of relaxation time as a function of the ratio of strengths for short- and long-range interactions is in contrast with that seen in the case of Watts-Strogatz (WS) small-world network model [22]. This reinforces previous results that the dynamics of WS networks are strikingly different from that of modular networks although their structural properties are similar [8]. It also suggests that a system may maintain diversity by using weak links between their constituent communities [17].

In this Rapid Communication we have shown that the order-disorder transition in an Ising model defined on a modular network shows the existence of three distinct phases. While the disordered and globally ordered phases are similar to those expected for a homogeneous network, the existence of a phase with local ordering within modules but no global order is a novel effect of the modular structure. This has significant implications for ordering dynamics in real systems, e.g., consensus formation on social networks. It indicates that, under certain conditions, homogeneous groups with contrary opinions can coexist forever even when mutual interactions between agents favor consensus so that society becomes polarized. Our results suggest that such tendencies can be countered by increasing intercommunity communication and improving the overall penetration of mass media in the society. Although the present Rapid Communication looks at the case of two competing choices, it is possible that the effects seen here extend to the case of agents choosing between multiple (>2) alternatives, e.g., in the context of q-state Potts spin dynamics on modular networks. Further, the network can be made more realistic from a social perspective by considering the presence of hubs and a broad link-strength distribution [9]. In summary, our results imply that due to mesoscopic structural inhomogeneity, equilibrium as well as dynamical properties of local regions in a complex network may depart significantly from those for the entire network.

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